HOMOTOPY 7 SPHERES

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In this note we want to use the calculation that the smooth structure set of $\mathbb{C}P^3$ is isomorphic to the integers to understand which homotopy 7-spheres can be built as an S^1-bundle over a fake $\mathbb{C}P^3$.

The main observation behind this is the following classification result, which is mainly due to Sullivan:

Theorem 1. *Let $n \geq 3$. Then the canonical map*

$$\mathcal{S}_{PL}(\mathbb{C}P^n) \longrightarrow \mathcal{N}_{PL}(\mathbb{C}P^{n-1})$$

is a bijection.

Moreover the splitting invariants *give an isomorphism*

$$\mathcal{N}_{PL}(\mathbb{C}P^n) \longrightarrow \prod_{i=2}^{n} L_{2i}(\mathbb{Z})$$

Corollary 2. *The PL-structure set of $\mathbb{C}P^3$ is isomorphic to the integers:*

$$\mathcal{S}_{PL}(\mathbb{C}P^3) \xrightarrow{\cong} \mathcal{N}_{PL}(\mathbb{C}P^2) \cong \mathbb{Z}$$

Now we can use smoothing theory to argue that all these PL-manifolds are actually uniquely smoothable: There is the homotopy fiber sequence

$$PL/O \longrightarrow BO \longrightarrow BPL$$

and PL/O is 6-connected due to results by Cerf. In particular for any 6-dimensional CW-complex X we have a bijection

$$[X, BO] \longrightarrow [X, BPL] \; .$$

Now smoothing theory tells us that the forgetful map

$$\mathcal{S}_{Diff}(\mathbb{C}P^3) \longrightarrow \mathcal{S}_{PL}(\mathbb{C}P^3)$$

is a bijection and so we obtain a countably infinite family of smooth manifolds homotopy equivalent to $\mathbb{C}P^3$.

We want to remark that they are pairwise non-homeomorphic, which follows most easily from the observation that Sullivan's theorem holds also in the topological category and we obtain a diagram

$$\begin{array}{ccc} \mathcal{S}_{PL}(\mathbb{C}P^n) & \xrightarrow{\cong} & \mathcal{N}_{PL}(\mathbb{C}P^{n-1}) \\ \downarrow & & \downarrow \\ \mathcal{S}_{TOP}(\mathbb{C}P^n) & \xrightarrow[\cong]{} & \mathcal{N}_{TOP}(\mathbb{C}P^{n-1}) \end{array}$$

in which the bottom map is also an isomorphism. Furthermore there is an isomorphism

$$\mathcal{N}_{TOP}(\mathbb{C}P^n) \longrightarrow \prod_{i=1}^{n} L_{2i}(\mathbb{Z})$$

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and the map from PL-normal invariants to topological normal invariants is given by

$$\prod_{i=2}^{n} L_{2i}(\mathbb{Z}) \longrightarrow \prod_{i=1}^{n} L_{2i}(\mathbb{Z})$$

$$(x_2, \ldots, x_n) \longmapsto (\mathrm{red}_2(x_2), x_2, \ldots, x_n)$$

This of course implies that the map

$$\mathcal{S}_{PL}(\mathbb{C}P^3) \longrightarrow \mathcal{S}_{TOP}(\mathbb{C}P^3)$$

is injective. This does not from the definition say that all fake complex projective spaces are pairwise non-homeomorphic, there is an issue with exotic self-homotopy equivalences. But since any self-homotopy equivalence of $\mathbb{C}P^3$ is homotopic to a homeomorphism this should imply the claim. A different proof which does not use the topological structure set is related to another observation we want to make. Up to now a smooth fake $\mathbb{C}P^3$ is classified by its second splitting invariant which is computed as follows:

Let $f\colon X \to \mathbb{C}P^3$ be a homotopy equivalence. Then we let $f\colon M \to \mathbb{C}P^2$ be the transverse inverse image of $\mathbb{C}P^2$ along f. It is standard that this will be a degree one normal map, and thus has a surgery obstruction, which is given by

$$\sigma(M, f) = \tfrac{\mathrm{sign}(M)-1}{8}.$$

It would be nicer to have a more direct invariant that also detects this fake complex projective space X and it turns out that the tangent bundle is such an invariant.

For this we observe that

$$[\mathbb{C}P^3, BO] \longrightarrow [\mathbb{C}P^2, BO] \xrightarrow{p_1} \mathbb{Z}$$

are both isomorphisms, such that the map

$$[\mathbb{C}P^3, BO] \longrightarrow \mathbb{Z}$$

$$E \longmapsto p(E)$$

is an isomorphism, where $p(E)$ is such that $p_1(E) = p(E) \cdot x^2$.

Now we have two maps from the smooth structure set to the integers:

$$\begin{array}{ccc} \mathcal{S}_{Diff}(\mathbb{C}P^3) & \xrightarrow{\Theta} & \mathbb{Z} \\ \downarrow \sigma & \nearrow_{\alpha} & \\ \mathbb{Z} & & \end{array}$$

where σ is the earlier described surgery obstruction of the associated normal invariant over $\mathbb{C}P^2$ and Θ is the map that sends X to $p(\tau_X)$ and we would like to understand the map α making this diagram commute.

For this we simply compute Θ as follows. We again consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{C}P^3 \\ \uparrow & & \uparrow \\ M & \xrightarrow[f]{} & \mathbb{C}P^2 \end{array}$$

We want to compute

$$p_1(\tau_X) \in H^4(X; \mathbb{Z})$$

but from the diagram we see that the map $j : M \to X$ induces an isomorphism on fourth cohomology, and using that $f\colon M \to \mathbb{C}P^2$ is degree one the number we want to calculate is

$$j^* p_1(\tau_X) = \Theta(X) \cdot \mu_M$$

where μ_M is such that $f^*(x^2) = \mu_M$.

Now we can use that

$$\begin{aligned} j^*p_1(\tau_X) &= p_1(j^*(\tau_X)) = p_1(\tau_M \oplus \nu_{M,X}) \\ &= p_1(\tau_M \oplus f^*(\gamma)) \\ &= p_1(\tau_M) + f^*(p_1(\gamma)) \\ &= 3\text{sign}(M)\cdot \mu_M + \mu_M \end{aligned}$$

where the last equation follows from the signature theorem for M and the fact that $p_1(\gamma) = x^2 \in H^4(\mathbb{C}P^2;\mathbb{Z})$.

This tells us that

$$\Theta(X) = 3\text{sign}(M) + 1 = 3(\text{sign}(M) - 1) + 4.$$

Hence if we let $\alpha : \mathbb{Z} \to \mathbb{Z}$ be the map $k \mapsto 24k + 4$ we see that

$$\alpha(\sigma(X)) = \alpha(\tfrac{\text{sign}(M)-1}{8}) = 3(\text{sign}(M) - 1) + 4$$

as claimed.

In particular we see in particular that Θ is injective and the fake complex projective spaces are classified by their first Pontrjagin class. Now using the theorem of Novikov that the rational Pontrjagin classes are homeomorphism invariants we obtain a different proof that the fake $\mathbb{C}P^3$'s are pairwise non-homeomorphic.

What we now want to study is the following. We use a homotopy theoretic construction over $\mathbb{C}P^3$ to obtain new a new manifold. Now we can apply the same construction to a fake $\mathbb{C}P^3$ because it only depended on the homotopy type. So we will obtain homotopy equivalent manifolds as outputs and then want to study them as elements in the structure set of the output over the standard $\mathbb{C}P^3$.

We start out with the tautological bundle over $\mathbb{C}P^3$. This is a principal S^1-bundle over $\mathbb{C}P^3$ classified by the canonical map $\mathbb{C}P^3 \to \mathbb{C}P^\infty$ and fits into a sequence

$$S^1 \longrightarrow S(\gamma) \longrightarrow \mathbb{C}P^3 \longrightarrow \mathbb{C}P^\infty\ .$$

It is of course know that $S(\gamma)$ is diffeomorphic to S^7. Now let $X \to \mathbb{C}P^3 \in \mathcal{S}_{Diff}(\mathbb{C}P^3)$. We obtain a commutative diagram

$$\begin{array}{ccccccc} S^1 & \longrightarrow & S(X) & \longrightarrow & X & \longrightarrow & \mathbb{C}P^\infty \\ \| & & \downarrow & & \downarrow \simeq & & \| \\ S^1 & \longrightarrow & S^7 & \longrightarrow & \mathbb{C}P^3 & \longrightarrow & \mathbb{C}P^\infty \end{array}$$

Hence it follows that

$$[S(X) \longrightarrow S^7] \in \mathcal{S}_{Diff}(S^7) \cong \Theta^7 \cong \mathbb{Z}/28\mathbb{Z}$$

and we will compute the image of this map

$$\begin{array}{ccc} \mathcal{S}_{Diff}(\mathbb{C}P^3) & \longrightarrow & \Theta^7 \\ X & \longmapsto & S(X) \end{array}$$

For this we will use an almost complex analogue of the usual Eells-Kuiper invariant for spin-7-manifolds with rational trivial spin-characteristic class.

Remark. We picked a specific bundle over $\mathbb{C}P^3$ which has the property that its first chern class is a generator. Clearly there are two such bundles, but they differ only in the complex structure, the underlying real vectorbundles are isomorphic and hence also the sphere bundles are diffeomorphic. Also, applying the Serre-spectral sequence to such an S^1-bundle over $\mathbb{C}P^3$ with primitive first chern class gives that the total space is a homotopy sphere.

First we recall the Eells-Kuiper invariant for homotopy 7-spheres, which is an isomorphism

$$\Theta^7 \xrightarrow{\ \mu\ } \mathbb{Z}/28\mathbb{Z}$$

given as follows.

Since $\mathrm{MSpin}_7 = 0$ it follows that given any $\Sigma \in \Theta^7$ there is a spin coboundary W for Σ. We then define

$$\mu_{\mathrm{spin}}(\Sigma) = \tfrac{p_W^2 - \mathrm{sign}(W)}{8} \bmod 28$$

where $p_W = \frac{1}{2}p_1(W)$ is the spin characteristic class of W. This is well-defined since for closed spin 8-manifolds X we have that

$$\tfrac{p_X^2 - \mathrm{sign}(X)}{8} = 28\hat{A}(X) \in 28\mathbb{Z} \tag{1}$$

Now we apply this to the closed manifold $X = \overline{W} \cup_\Sigma W'$ obtained by glueing two coboundaries of Σ together along the common boundary. We then know

$$0 = \tfrac{p_X^2 - \mathrm{sign}(X)}{8} = -\tfrac{p_W^2 - \mathrm{sign}(W)}{8} + \tfrac{p_{W'}^2 - \mathrm{sign}(W')}{8} \bmod 28$$

and obtain that the value $\mu_{\mathrm{spin}}(\Sigma)$ is independent of the choice of coboundary W.

To show that this map is a bijection we argue as follows. On the one hand we know that $bP^8 = \Theta^7$, and furthermore $bP^8 \cong \mathbb{Z}/28\mathbb{Z}$ is computed precisely by a parallelizable coboundary and taking its signature divided by 8 mod 28. Hence from the fact that μ_{spin} is independent of the coboundary it suffices to see that for a parallelizable coboundary W (which is of course in particular spin) we get

$$\mu_{\mathrm{spin}}(\Sigma) = -\tfrac{\mathrm{sign}(W)}{8} \bmod 28$$

which is clear: the assumption that W is parallelizable implies $p_W = 0$.

Hence, the main idea is, that one can calculate the Eells-Kuiper invariant of a homotopy sphere also in terms of coboundaries that are not parallelizable, as soon as we can establish a relation of the sort (1).

We have seen previously that fake $\mathbb{C}P^3$'s are parametrized by $k \in \mathbb{Z}$, so let us denote the fake $\mathbb{C}P^3$ with first Pontrjagin class $(24k+4)x^2$ by $X(k)$.

In this notation we are interested in calculation the Eells-Kuiper invariant of $S(X(k))$, the total space of the canonical S^1-bundle over $X(k)$. These are canonically the boundary of the disk bundles $D(X(k))$. We denote by $p : D(X(k)) \to X(k)$ the projection. Of course it is a homotopy equivalence, hence it follows easily that since the tautological S^1-bundle over $X(k)$ is not a spin bundle $(c_1(\gamma) = x)$ but $\mathbb{C}P^3$ is spin, the disk bundle itself is *not* a spin manifold.

Hence we need a different way to compute the Eells-Kuiper invariant of $S(X(k))$ in terms of the algebraic topology of $D(X(k))$. For this we will consider a variant of the Eells-Kuiper invariant for almost complex manifolds.

Lemma 3. *Let X be an almost complex closed 8 manifold. Then we have*

$$28\mathrm{td}(X) = \frac{3c_2^2 + 4c_2c_1^2 - c_1^4 - 3\mathrm{sign}(X)}{24}.$$

Using the same reasoning as in the spin case, we obtain a well-defined invariant for any homotopy sphere Σ with (stably) almost complex coboundary X as follows

$$\mu_{ac}(\Sigma) = \frac{3c_2^2 + 4c_2c_1^2 - c_1^4 - 3\mathrm{sign}(X)}{24} \bmod 28.$$

Lemma 4. *For any homotopy sphere Σ we have that*

$$\mu_{\mathrm{spin}}(\Sigma) = \mu_{ac}(\Sigma).$$

Proof. This follows again from the fact that it is independent of the coboundary. Hence we are free to choose a parallelizable coboundary, in which case both formulas reduce to to signature divided by 8. □

Remark. It is worthwhile to consider what happens for manifolds X that are both almost complex and spin. This amounts to saying that there is a class $c \in H^2(X;\mathbb{Z})$ such that $2c = c_1$. Using this it is not complicated to see that one has

$$p_X^2 = 3c_2^2 + 4c_2c_1^2 - c_1^4 \bmod 28.$$

Having this we want to calculate the Eells-Kuiper invariant of $S(X(k))$ in terms of this almost complex invariant just described, therefore we need to define an almost complex structure on $D(X(k))$ and calculate its chern classes.

For this, we first explain how to obtain an almost complex structure on $X(k)$ and calculate their chern classes.

From the diagram

$$\begin{array}{ccc} [\mathbb{C}P^3, BO] & \xrightarrow{\cong} & [\mathbb{C}P^2, BO] \\ \uparrow & & \uparrow \\ [\mathbb{C}P^3, BU] & \longrightarrow & [\mathbb{C}P^2, BU] \end{array}$$

the fact that $p_1 : [\mathbb{C}P^2, BO] \to \mathbb{Z}$ is an isomorphism, and the fact that the bottom horizontal arrow is split surjective, it follows that it suffices to construct a function $\alpha\colon \mathbb{Z} \to [\mathbb{C}P^2, BU]$ with the property that $p_1(\alpha(k)) = 24k + 4 = p_1(X(k))$.

Now, one computes $[\mathbb{C}P^2, BU]$ as follows. There is a short exact sequence

$$0 \longrightarrow \pi_4(BU) \longrightarrow [\mathbb{C}P^2, BU] \longrightarrow \pi_2(BU) \longrightarrow 0$$

and a splitting is given by noting that the generator of $\pi_2(BU)$ is the tautological bundle γ over $S^2 = \mathbb{C}P^1$ which of course canonically comes from the tautological bundle over $\mathbb{C}P^2$, which we also denote by γ. This implies that any bundle $E \in [\mathbb{C}P^2, BU]$ is of the form

$$E = m \cdot \tau_{\mathbb{H}} + n \cdot \gamma.$$

We thus have the following formulas for such an E

(i) $c_1(E) = m \cdot c_1(\tau_{\mathbb{H}}) + n \cdot c_1(\gamma) = nx$,
(ii) $c_2(E) = c_2(m\tau_{\mathbb{H}}) + c_2(n\gamma) = mx^2 + \frac{n(n-1)}{2}x^2$, and
(iii) $p_1(E) = (n - 2m)x^2$.

Hence if we define α by the formula

$$k \mapsto 12k \cdot \tau_{\mathbb{H}} + 4\gamma$$

we obtain

$$p_1(\alpha(k)) = 24k + 4$$

which just means that the bundle

$$-12k\tau_{\mathbb{H}} + 4\gamma \in [X(k), BU]$$

is an almost complex structure on the manifold $X(k)$. Summarizing we get that in terms of this specific almost complex structure the chern classes are given by

(i) $c_1(X(k)) = 4x$, and
(ii) $c_2(X(k)) = (-12k + 6)x^2$.

Next we have to calculate the chern classes of the manifold $D(X(k))$. The tangent bundle has the formula

$$\tau_{D(X(k))} = p^*(\tau_{X(k)} \oplus \gamma)$$

and hence we get that

$$c_1(D(X(k)) = 5x$$

and

$$\begin{aligned} c_2(D(X(k)) &= c_2(X(k)) + c_1(X(k)) \cdot c_1(\gamma) \\ &= (-12k + 6)x^2 + 4x^2 = (-12k + 10)x^2 \end{aligned}$$

Furthermore we remark, that calculating in the relative cohomology of the pair $(D(X(k)), S(X(k)))$ is the same as calculating in the quotient, which is the Thom space of the bundle γ over $X(k)$. Since $X(k)$ is homotopy equivalent to $\mathbb{C}P^3$ it follows that this Thom space in question is homotopy equivalent to $\mathbb{C}P^4$ and thus the intersection form is the standard form over $\mathbb{Z}$.

We can hence calculate the Eells-Kuiper invariant of $S(X(k))$ as follows

$$\begin{aligned}
\mu_{ac}(S(X(k))) &= \frac{3c_2^2 + 4c_2c_1^2 - c_1^4 - 3\mathrm{sign}(D(X(k))}{24} \bmod 28 \\
&= \frac{3(-12k+10)^2 + 4(-12k+10)25 - 625 - 3}{24} \bmod 28 \\
&= \frac{3(144k^2 - 240k + 100) - 1200k + 1000 - 625 - 3}{24} \bmod 28 \\
&= \frac{432k^2 - 720k + 300 - 1200k + 1000 - 625 - 3}{24} \bmod 28 \\
&= 18k^2 + 4k \bmod 28
\end{aligned}$$

The values this function takes is given by

$$\mathrm{Im}(\mu) = \{0, 6, 8, 10, 14, 20, 22, 24\}.$$

We can compare this to the Eells-Kuiper invariant of S^3-bundles over S^4 that were first studied by Milnor. In [Eells-Kuiper] it is calculated that the values that occur there are precisely given by

$$\{0, 1, 3, 6, 7, 8, 10, 13, 14, 15, 17, 20, 21, 22, 24, 27\}$$

thus we obtain the following

Corollary 5. *The homotopy* 7*-spheres that arise as* S^1*-bundles over a fake complex projective* 3*-space are precisely the total spaces of* S^3*-bundles over* S^4 *with euler number one and even Eells-Kuiper invariant.*

One could try to use this fact to give a somewhat concrete construction of fake complex projective spaces in dimension three.

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