SMOOTH NORMAL INVARIANTS OF $\mathbb{C}P^2$

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In this article we want to prove the following

Theorem 1. *The group of smooth normal invariants of* $\mathbb{C}P^2$ *is given by*

$$\mathcal{N}^{sm}(\mathbb{C}P^2) \cong \mathbb{Z}.$$

Proof. To compute this group up to an extension problem, we recall that smooth normal invariants of a manifold M are given by homotopy classes of maps to G/O:

$$[M, G/O] \cong \mathcal{N}^{sm}(M).$$

We now consider the cofibration sequence

$$S^3 \xrightarrow{\eta} S^2 \longrightarrow \mathbb{C}P^2 \xrightarrow{c} S^4 \longrightarrow S^3$$

which gives a short exact sequence

$$0 \longrightarrow \pi_4(G/O) \xrightarrow{c^*} [\mathbb{C}P^2, G/O] \longrightarrow \pi_2(G/O) \longrightarrow 0$$

because $\pi_3(G/O) = 0$. We recall that $\pi_4(G/O) \cong \mathbb{Z}$ and $\pi_2(G/O) \cong \mathbb{Z}/2\mathbb{Z}$, so we want to show that this sequence does not split.

We now consider the following maps. Let $\alpha \in \pi_4(G/O)$ be some element. Since $\pi_4(G/O) \cong \mathcal{N}^{sm}(S^4)$ this corresponds to a bordism class of a degree one normal map

$$M \xrightarrow{f} S^4$$

where M is a smooth 4-manifold. The fact that this map is normal implies that M is spin, and so we see that the signature $\text{sign}(M)$ is divisible by 16, according to Rokhlin's theorem. Moreover since the signature is a bordism invariant we have a well-defined map

$$\begin{array}{ccc} \pi_4(G/O) & \xrightarrow{\quad\sigma\quad} & \mathbb{Z} \\ [f : M \to S^4] & \longmapsto & \frac{\text{sign}(M)}{16} \end{array} .$$

Actually, before dividing by 16 this is just the simply-connected surgery obstruction, which is still well-defined in these low dimensions (it being zero just does not imply that a degree one normal map can be improved by a series of surgeries to a homotopy equivalence anymore).

Similary we can take an element $\beta \in [\mathbb{C}P^2, G/O]$ and represent it by a degree one normal map

$$N \xrightarrow{g} \mathbb{C}P^2$$

The simply connected surgery obstruction here is given by the difference of the signatures, which is always divisible by 8 (on the surgery kernel, the intersection form is even), hence we obtain a well-defined map

$$\begin{aligned} [\mathbb{C}P^2, G/O] &\xrightarrow{\hat{\sigma}} \mathbb{Z} \\ [g : N \to \mathbb{C}P^2] &\longmapsto \tfrac{\mathrm{sign}(N)-1}{8} \end{aligned}$$

We now claim that the diagram

$$\begin{array}{ccc} \pi_4(G/O) & \xrightarrow{c^*} & [\mathbb{C}P^2, G/O] \\ \sigma\downarrow & & \downarrow\hat{\sigma} \\ \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} \end{array}$$

commutes, i.e. we want to show that if

$$c^*[M \to S^k] = [N \to \mathbb{C}P^2]$$

then we get

$$\mathrm{sign}(M) = \mathrm{sign}(N) - 1 \tag{1}$$

For this we will show how to compute the signature difference of a normal invariant in terms of the corresponding map to G/O.

So we start with a map $f : M \to G/O$, where M is some smooth manifold. We denote by E the bundle that is classified by the composite

$$M \xrightarrow{f} G/O \longrightarrow BO$$

The normal invariant corresponding to the map f is constructed as follows. The fact that E has a lift to G/O gives a proper fiber homotopy equivalence

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & M \times \mathbb{R}^k \\ \pi\downarrow & & \downarrow \\ M & = & M \end{array}$$

We let N be the transverse inverse image of M in $M\times\mathbb{R}^k$ along Φ and $\varphi = \pi|_N : N \to M$. This is a degree one map since Φ is a proper fiber homotopy equivalence. Moreover we can compute the stable tangent bundle of N as follows:

$$\begin{aligned} \tau_N = \tau_N \oplus \nu_{N,E} &= (\tau_E)|_N \\ &= i^*(\pi^*(\tau_M \oplus E)) \\ &= \varphi^*(\tau_M \oplus E) \end{aligned}$$

This is because $\nu_{N,E}$ is trivial as the normal bundle of M in $M \times \mathbb{R}^k$ is trivial. In particular the map φ is normal with respect to the reference bundle $\nu_M - E$.

We now compute the signature difference as follows:

$$\begin{aligned}\operatorname{sign}(N)-\operatorname{sign}(M) &= \langle L(\tau_N),[N]\rangle - \langle L(\tau_M),[M]\rangle \\ &= \langle L(\varphi^*(\tau_M\oplus E)),[N]\rangle - \langle L(\tau_M),[M]\rangle \\ &= \langle \varphi^*(L(\tau_M\oplus E)),[N]\rangle - \langle L(\tau_M),[M]\rangle \\ &= \langle L(\tau_M\oplus E),[M]\rangle - \langle L(\tau_M),[M]\rangle\end{aligned}$$

Now in general we don't know how to continue this computation because the L-polynomial does not behave well with respect to Whitney sums of bundles. But in the case that M is an orientable 4-manifold actually the L-polynomial is equal to $\frac{1}{3}\cdot p_1$, so is in particular additive. Thus assuming that M is of dimension 4 we can continue to compute

$$\begin{aligned}\operatorname{sign}(N)-\operatorname{sign}(M) &= \langle L(\tau_M\oplus E),[M]\rangle - \langle L(\tau_M),[M]\rangle \\ &= \langle L(\tau_M),[M]\rangle + \langle L(E),[M]\rangle - \langle L(\tau_M),[M]\rangle \\ &= \langle L(E),[M]\rangle \\ &= \frac{1}{3}\cdot\langle p_1(E),[M]\rangle\end{aligned}$$

i.e. the signature difference can be computed in terms of the bundle classified by the map to G/O.

Now let us assume we have a map $\varphi : M' \to M$ of smooth 4-manifolds and let $f : M \to G/O$ be a normal invariant. We again denote by E the stable vector bundle classified by f. We denote by $\operatorname{sign}(f)$ the simply connected surgery obstruction of f, i.e. the signature difference of the normal invariant given by f. We have just computed that

$$\operatorname{sign}(f) = \frac{1}{3}\cdot\langle p_1(E),[M]\rangle.$$

The same formula is of course true for the map $f\circ\varphi$, hence we can compute

$$\begin{aligned}\operatorname{sign}(f\circ\varphi) &= \tfrac{1}{3}\cdot\langle p_1(\varphi^*(E)),[M']\rangle \\ &= \tfrac{1}{3}\cdot\langle p_1(E),\varphi_*[M']\rangle \\ &= \tfrac{\deg(\varphi)}{3}\cdot\langle p_1(E),[M]\rangle \\ &= \deg(\varphi)\cdot\operatorname{sign}(f)\end{aligned}$$

In particular, if the map φ has degree one, then the signature difference does not change.

Applying this to the map $c : \mathbb{C}P^2 \to S^4$ implies the claimed equation (1)

$$\operatorname{sign}(M) = \operatorname{sign}(N) - 1.$$

Next we claim that the map $\sigma : \pi_4(G/O) \to \mathbb{Z}$ is an isomorphism, i.e. that there exists a smooth manifold of signature 16 with a degree one normal map to S^4. For this we can take any $K3$ surface, for instance the classical Kummer surface. It is a spin manifold of signature -16 (the sign of course does not matter for this business) and since all spin manifolds of dimension four are almost parallelizable it admits a degree one normal map to the sphere.

We now claim that there exists a smooth manifold of signature 9, which admits a degree one normal map to $\mathbb{C}P^2$. If this is true, then the we can show that the extension computing $[\mathbb{C}P^2, G/O]$ is non-trivial as follows. We then know that the map

$$\hat{\sigma} : [\mathbb{C}P^2, G/O] \longrightarrow \mathbb{Z}$$

is onto. So if we assume that the sequence splits, the restriction of $\hat{\sigma}$ to the $\mathbb{Z}$ summand would be an isomorphism. But using the commutativity of the above square involving the map $\sigma : \pi_4(G/O) \to \mathbb{Z}$ we know that this map is multiplication by 2, which is not an isomorphism, a contradiction.

So we are left to show that there exists a smooth manifold of signature 9 with a degree one normal map to $\mathbb{C}P^2$.

For this we first consider the following general

Lemma *Let M be a smooth, closed, connected, oriented 4-manifold. Then homotopy classes of degree one maps $f : M \to \mathbb{C}P^2$ are in bijection to the set*

$$\left\{ z \in H^2(M;\mathbb{Z}) : z^2 = \mu_M \right\}$$

where μ_M is determined by the property $\langle \mu_M, [M] \rangle = 1$.

Proof. By cellular approximation we have $[M, \mathbb{C}P^2] \cong [M, \mathbb{C}P^\infty]$. Now being of degree one means that

$$f^*(\mu_{\mathbb{C}P^2}) = \mu_M$$

but $\mu_{\mathbb{C}P^2} = x^2$ where $x \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ denotes a polynomial generator. From this it follows that $z = f^*(x)$ fulfills $z^2 = \mu_M$. The argument is clearly reversible. □

We now want to understand how to detect that a given degree one map $M \to \mathbb{C}P^2$ is normal. For this we recall the following

Lemma *For an oriented closed 4-manifold M, the map*

$$[M, BSO] \xrightarrow{w_2 \times p_1} H^2(M;\mathbb{Z}/2\mathbb{Z}) \times H^4(M;\mathbb{Z})$$

is an injective group homomorphism.

Proof. It is a group homomorphism since w_1 is trivial for BSO bundles by definition, and by assumption p_1 lives in a torsionfree group, hence in both cases the additivity follows from the Cartan formula. The injectivity follows from Postnikov theory of BSO and the according extension problem as follows. Again by cellular approximation we have that

$$[M, BSO] \cong [M, P_5(BSO)]$$

and the computation of the low-dimensional homotopy groups of BSO tells us that there is a fibration sequence

$$K(\mathbb{Z}, 4) \longrightarrow P_5(BSO) \xrightarrow{w_2} K(\mathbb{Z}/2\mathbb{Z}, 2)$$

which, by mapping into this fibration gives an exact sequence

$$H^1(M;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^4(M;\mathbb{Z}) \longrightarrow [M, BSO] \xrightarrow{w_2} H^2(M;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^5(M;\mathbb{Z})$$

Now since $H^4(M;\mathbb{Z})$ is torsionfree and $H^5(M;\mathbb{Z}) = 0$ we get a short exact sequence

$$0 \longrightarrow H^4(M;\mathbb{Z}) \longrightarrow [M, BSO] \xrightarrow{w_2} H^2(M;\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

Now we claim that the composition

$$H^4(M;\mathbb{Z}) \longrightarrow [M, BSO] \xrightarrow{p_1} H^4(M;\mathbb{Z})$$

is injective, which implies the claim of the lemma. To prove this we compare this to the case of the 4-sphere by a degree one map $M \to S^4$ to get

$$\begin{array}{ccccc} H^4(M;\mathbb{Z}) & \rightarrowtail & [M, BSO] & \xrightarrow{p_1} & H^4(M;\mathbb{Z}) \\ \cong\uparrow & & \uparrow & & \uparrow\cong \\ H^4(S^4;\mathbb{Z}) & \rightarrowtail & \pi_4(BSO) & \xrightarrow[p_1]{} & H^4(S^4;\mathbb{Z}) \end{array}$$

It is a standard fact that the map $p_1 : \pi_4(BSO) \to H^4(S^4;\mathbb{Z})$ is injective (indeed it can be identified with multiplication by 2 on the integers). Hence the claim follows. ☐

We thus get the following

Corollary *Suppose M is as above and $f : M \to \mathbb{C}P^2$ is a degree one map. Then suppose there is a bundle E over $\mathbb{C}P^2$ such that*

$$f^*(p_1(E)) = p_1(\nu_M) \text{ and } f^*(w_2(E)) = w_2(\nu_M)$$

then f is indeed a degree one normal map (with respect to the reference bundle E).

We are now ready to construct the normal invariant of $\mathbb{C}P^2$ with signature 9 as follows.

We consider the manifold $M = \mathcal{E}\sharp\mathbb{C}P^2$ where $\mathcal{E}$ denotes some Enriques surface. The key feature of these complex surfaces are the following

- the first chern class $c_1(\mathcal{E})$ is a torsion element,
- the universal covering of $\mathcal{E}$ is a $K3$ surface, in particular we have $\text{sign}(\mathcal{E}) = 8$ (we choose the opposite orientation of the canonical complex orientation)

Of course, by design this implies that $\text{sign}(M) = 9$. With this at hand we consider the degree one map given under the above correspondence by the element

$$(c_1(\mathcal{E}), x) \in H^2(\mathcal{E};\mathbb{Z}) \oplus H^2(\mathbb{C}P^2;\mathbb{Z}) \cong H^2(\mathcal{E}\sharp\mathbb{C}P^2;\mathbb{Z})$$

where again x denotes the polynomial generator. This is degree one since $c_1(\mathcal{E})^2 = 0$ by the property that $c_1(\mathcal{E})$ is torsion.

Next we argue that there is a bundle making this map a degree one *normal* map. First let us assume that there is a bundle E over $\mathbb{C}P^2$ such that the diagram

$$\begin{array}{ccc} \nu_M & \longrightarrow & E \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & \mathbb{C}P^2 \end{array}$$

is a pullback diagram. Then by the signature formula of Hirzebruch we get

$$\begin{aligned} 27 = 3 \cdot \operatorname{sign}(M) &= -\langle p_1(\nu_M), [M]\rangle \\ &= -\langle p_1(E), [\mathbb{C}P^2]\rangle \end{aligned}$$

Moreover it is a standard computation that

$$\begin{array}{ccc} [\mathbb{C}P^2, BO] & \longrightarrow & \mathbb{Z} \\ E & \longmapsto & \langle p_1(E), [\mathbb{C}P^2]\rangle \end{array}$$

is an isomorphism. Hence we can consider the bundle E over $\mathbb{C}P^2$ with pontrjagin number -27. By the previous lemma to see wether the map f is normal it now suffices to check that

$$f^*(w_2(E)) = w_2(\nu_M) = w_2(\tau_M)$$

For this we note that $w_2(E)^2 = \operatorname{red}_2(p_1(E)) = \operatorname{red}_2(x)^2$ which forces $w_2(E) = \operatorname{red}_2(x)$. Hence by design of the map f we get that

$$f^*(w_2(E)) = \operatorname{red}_2(c_1(E), x) = w_2(M)$$

as needed. □