Bonn International Graduate School in Mathematics

K- and L-Theory of complex C^*-Algebras

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C^*-Algebras and K-Theory

The main objects of this poster are invariants attached to C^*-algebras - the main object of study in what is nowadays called *non-commutative* topology.

Definition 1. A (complex) C^*-algebra A is a (complex) Banach algebra A together with an involution $x \mapsto x^*$ on A that fulfills the so-called C^*-identity

$$\|x^*x\| = \|x\|^2 \text{ for all } x \in A.$$

A *-homomorphism $\varphi\colon A \to B$ between C^*-algebras is a just morphism of algebras that commutes with the involutions, i.e. that for all $x \in A$ we have

$$\varphi(x^*) = \varphi(x)^*.$$

We denote the category of C^*-algebras by $\mathcal{C}^*$.

This seemingly harmless equation has strong implications to the structure of such algebras. For instance we have that

(1) Every *-homomorphism is automatically continouous,

(2) every injective *-homomorphism is automatically isometric, in particular

(3) the image of every *-homomorphism is automatically closed,

(4) the topology of A can be recovered from the underlying involutive ring.

Topology relates to C^*-algebras in the following fashion. Consider a space X with an action of a (to make life easier) discrete group G on it. We can associate to this pair a so-called *crossed-product* C^*-algebra $C_0(X) \rtimes G$ that is in some sense to be thought of as a C*-algebra of the *homotopy orbits* of the G-action on X. This is made precise by the following

Theorem (Green)**.** *Suppose the action of G on X is proper and free, then $C_0(X) \rtimes G$ and $C_0(X/G)$ are Morita equivalent.*

If G is the trivial group then $C_0(X) \rtimes G = C_0(X)$ so we recover the usual topology in C^*-algebras. On the other hand if the space X consists only of a point, we call the resulting C^*-algebra the *group C^*-algebra* and denote it by C^*G. It is a completion of the complex group ring $\mathbb{C}G$.

Definition 2. Associated to a C^*-algebra is a topological K-theory *spectrum* $\mathbb{K}^{\mathrm{top}}(A)$ whose homotopy groups are the so-called *topological K-theory groups* $K_*^{\mathrm{top}}(A)$ of A.

For example for a compact space one has an isomorphism

$$K_*^{\mathrm{top}}(C(X)) \cong K^*(X),$$

whereas studying the topological K-theory groups of the group C^*-algebra C^*G is the content of the *Baum-Connes conjecture* which predicts a certain map

$$\mathbb{K}_*^G(\underline{E}G) \longrightarrow K_*^{\mathrm{top}}(C^*G)$$

to be an isomorphism (if one interprets the group algebra in the correct way).

Surgery and L-Theory

Next we want to introduce a different invariant which is attached to involutive rings (and thus in particular to C^*-algebras). For this we want to recall the basic ideas of *surgery theory*. One of the main questions surgery theory was developed to answer is as follows:

Question. *Given a (say compact) CW-complex X, is X homotopy equivalent to a closed manifold?*

The main answer is to observe the following necessary condition: If X is homotopy equivalent to a closed manifold, then the cohomology of X must satisfy Poincaré duality. A refined version of this statement about the cohomology of X implies that X is what is called a *Poincaré complex*. Such spaces have a canonical spherical fibration associated to them, we call this spherical fibration by $\mathrm{SF}(X)$. If X is homotopy equivalent to a closed manifold M it follows that $\mathrm{SF}(X)$ is the underlying sphere bundle of the (stable) normal bundle ν_M of that manifold. In particular $\mathrm{SF}(X)$ must have a reduction to a vector bundle if X is homotopy equivalent to a closed manifold. If such a reduction exists, the possible reductions are (non-canonically) in bijection to the set of homotopy classes

$$[X, G/O].$$

This set is called the *set of normal invariants* of X, because elements of it can be represented by bordism classes of degree one maps $M \xrightarrow{f} X$ with the property that there is a vector bundle E over X such that $f^*(E) \cong \nu_M$ and M being a closed manifold. The following is one of the main theorems of surgery theory.

Theorem. *There exists an abelian group $L_*(\pi)$ which depends only on the fundamental group $\pi = \pi_1(X)$ of X and a map*

$$[X, G/O] \xrightarrow{\Theta} L_*(\pi)$$

called the surgery obstruction map*, such that if*

$$\Theta[M \xrightarrow{f} X] = 0$$

it follows that X is homotopy equivalent to a manifold M', and M' is normally bordant to M.

These obstruction groups are called L-groups and where first constructed geometrically by Wall. Later there was a purely algebraic definition of the L-groups $L_*(R,\tau)$ for any involutive ring (R,τ) using non-degenerate forms over R and a variant of those which are called formations (and are closely related to the automorphisms of non-degenerate forms). In this picture $L_*(\pi) = L_*(\mathbb{Z}\pi, w)$, where w denotes an involution which reflects the possible non-orientability of X.
Due to work of Ranicki, there is a functor from the category of involutive rings to the category of spectra,

$$\mathrm{Rings}^{\mathrm{inv}} \xrightarrow{\mathbb{L}} \mathrm{Sp}$$
$$(R,\tau) \longmapsto \mathbb{L}(R,\tau)$$

such that the homotopy groups of $\mathbb{L}(R,\tau)$ are the above described L-groups.
Notice that C^*-algebras are in particular involutive rings, and as such there are L-theory spectra and L-theory groups associated to C^*-algebras.
Here is a remarkable connection between the L-groups and the topological K-groups of a C^*-algebra.

Integral Maps of Spectra

Theorem (Miller)**.** *Let A be a complex C^*-algebra. Then there is a natural isomorphism*

$$K_*^{\mathrm{top}}(A) \xrightarrow{\cong} L_*(A)\,.$$

It relies on the fact, that every finitely generated projective module over a C^*-algebra has a canonical (up to homotopy) non-degenerate symmetric form over itself. A natural reflex of a topologist would be to ask the following

Question. *Does this isomorphism lift to an equivalence of functors with values in spectra?*

Let us denote (to have the usual notations from topology) $\mathrm{KU} = \mathbb{K}^{\mathrm{top}}(\mathbb{C})$ and we write $L\mathbb{C}$ for $\mathbb{L}(\mathbb{C})$, moreover we denote by ku and $\ell\mathbb{C}$ the connective covers of KU and $L\mathbb{C}$ respectively.
It is known that the spectra KU and $L\mathbb{C}$ are not equivalent, and we have the following stronger version that answers the question to the negative:

Theorem. *Any map $\mathrm{KU} \to L\mathbb{C}$, $L\mathbb{C} \to \mathrm{KU}$ and $\ell\mathbb{C} \to \mathrm{ku}$ induce zero on π_0. The space of maps $\mathrm{ku} \to L\mathbb{C}$ is interesting and will be investigated later.*

Proof. This boils down to some well-know properties in stable homotopy theory. The main ingredients are that π_0 of all spectra involved is isomorphic to the integers, so it remains to verify that maps induce zero after localizing at the prime (2). Now since L-spectra are all modules over the ring spectrum $\mathbb{L}(\mathbb{Z})$ and this is an algebra over MSO, it follows that all L-spectra in question (2)-locally split as wedges of Eilenberg-MacLane spectra. Thus it suffices to show the theorem if we replace all L-spectra by $\mathrm{H}\mathbb{Z}_{(2)}$. Then the crucial property is that the spectrum $\mathrm{KU} \wedge \mathrm{H}\mathbb{Z}$ is rational (which follows essentially because the multiplicative and the additive formal group law are isomorphic and the Bott element is invertible). It is important to not make the mistake to think that also $\mathrm{ku} \wedge \mathrm{H}\mathbb{Z}$ is rational (we made this mistake and got very confused about it), which explains why maps $\mathrm{ku} \to L\mathbb{C}$ can be non-trivial on π_0. □

Our goal will be to calculate the space of natural transformations $\mathrm{ku} \to \mathbb{L}$ from connective topological K-theory spectra to L-theory spectra of C^*-algebras. The main idea will be to calculate the set of natural transformations on the level of the abelian groups and hope to lift this to a homotopical setting, where spectra are available. For this we need to recall the following universal property of the KK-category.

Theorem (Higson,Uuye)**.** *The canonical functor $\mathcal{C}^* \to \mathrm{KK}$ has the following universal property: The induced functor*

$$\mathrm{Fun}(\mathrm{KK}, \mathrm{Ab}) \longrightarrow \mathrm{Fun}(\mathcal{C}^*, \mathrm{Ab})$$

is fully-faithful and has image the functors that send KK-equivalences to isomorphisms. Moreover those are exactly the functors that are homotopy invariant, split exact, *and* stable*. Furthermore topological K-theory satisfies these properties and becomes corepresentable on* KK*, and a corepresenting object is $\mathbb{C}$, i.e. $\mathbb{K}^{\mathrm{top}}(A) \cong \mathrm{KK}(\mathbb{C}, A)$.*

Using the theorem of Miller, one sees that L-theory satisfies all properties to factor over KK. By the previous theorem we hence obtain as an easy application of the Yoneda lemma that

$$\mathrm{Nat}(K^{\mathrm{top}}, L) \cong L_0(\mathbb{C}) \cong \mathbb{Z}$$

and it follows formally that Millers transformation corresponds to a unit in $\mathbb{Z}$.

A possible homotopical lifting

We want to try to lift this argument using a homotopical enhancement of the category KK. The first observation to obtain this is the following

Theorem (Uuye)**.** *The category $\mathcal{C}^*$ admits the structure of a fibration category, with weak equivalences given by the KK-equivalences. In particular there exists an ∞-category KK_∞ whose homotopy category is* KK *and the canonical map $\mathcal{C}^* \to \mathrm{KK}_\infty$ makes the diagram*

$$\begin{array}{ccc} \mathcal{C}^* & \longrightarrow & \mathrm{KK}_\infty \\ & \searrow & \downarrow \\ & & \mathrm{KK} \end{array}$$

For technical reasons we need the following

Lemma. *The ∞-category KK_∞ is stable, hence in particular enriched in the ∞-category Sp_∞ of spectra.*

We have the analogue to the theorem of Higson, by $\mathcal{N}(\mathcal{C}^*)$ we denote the nerve of the category $\mathcal{C}^*$ and thus view it as an ∞-category.

Theorem. *The induced functor of $\mathcal{N}(\mathcal{C}^*) \to \mathrm{KK}_\infty$ is an ∞-categorical localization, i.e.,*

$$\mathrm{Fun}(\mathrm{KK}_\infty, \mathrm{Sp}_\infty) \longrightarrow \mathrm{Fun}(\mathcal{N}(\mathcal{C}^*), \mathrm{Sp}_\infty)$$

is fully faithful and has image functors that are pointed and send KK-equivalences to equivalences of spectra.

Similar as to before $\mathbb{K}^{\mathrm{top}}$-theory will factor over KK_∞ and become corepresented by $\mathbb{C}$. The idea now is to show that $\mathbb{L}$-theory (in a suitable sense) factors over KK_∞. An ∞-categorical version of the Yoneda lemma should then allow to compute the space of natural transformations as wanted. This is work in progress.

References

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