Analytical Assembly and Index Theory

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The Baum-Connes Conjecture à la Kasparov

Let G be a countable discrete group. There is a map called the *analytical assembly map*

$$\mathcal{A} : RK_*^G(\underline{E}G) \longrightarrow K_*(C_r^*G)$$

between the G-equivariant compactly supported analytic K-homology of the classifying space for proper actions and the K-theory of the reduced group C^*-algebra. We recall that a classifying space for proper actions is a G-CW-complex whose H-fixed points are contractible if H is a finite subgroup of G and are empty else. Such a space is automatically a terminal object in the full subcategory of the equivariant homotopy category consisting of proper G-spaces.
The analytic assembly map turns up in the Baum-Connes conjecture:

Conjecture. *For every countable discrete group G the analytical assembly map*

$$\mathcal{A} : RK_*^G(\underline{E}G) \xrightarrow{\cong} K_*(C_r^*G)$$

is an isomorphism of abelian groups.

There are many definitions of the assembly map and one of my goals is to show that all these constructions indeed coincide. Here I will only talk about two descriptions - one purely in terms of KK-theory and an index-theoretic description. The following (analytical) construction of the assembly map is the approach proposed by Kasparov, and is the one used when applying the so-called Dirac-dual-Dirac method (a strategy of proving the Baum-Connes conjecture). To define this map we need the following

Lemma. *Let X be a proper and cocompact G-space. Then there exists a function $\psi \in C_c(X)$ such that for all $x \in X$ we have $\sum_{g \in G} \psi(gx) = 1$. Moreover the element*

$$p_X(g,x) = \sqrt{\psi(x) \cdot \psi(g^{-1}x)} \in C_0(X) \rtimes G$$

is a projection and its K-theory class $[p_X] \in KK(\mathbb{C}, C_0(X) \rtimes G)$ is independent of the function ψ.

The following proposition is a classical fact in KK-theory and was first formulated and proven in [1].

Proposition. *There is a functor $j_r^G : KK^G \longrightarrow KK$ from the equivariant KK-category to the unequivariant KK-category, called the reduced descent homomorphism. On objects this functor is given by the reduced crossed product functor, i.e., for any G-C^*-algebra A we have*

$$j_r^G(A) = A \rtimes_r G.$$

Sometimes reduced and full crossed products by G coincide. A result concerning this is the following

Proposition. *If X is a proper G-space then $C_0(X) \rtimes G = C_0(X) \rtimes_r G$. Moreover this algebra is nuclear.*

Using this we can now define the analytical assembly map by the map induced on colimits of the following composite

$$KK_*^G(C_0(X), \mathbb{C}) \xrightarrow{j_r^G} KK_*(C_0(X) \rtimes G, C_r^*G) \xrightarrow{-\circ[p_X]} KK_*(\mathbb{C}, C_r^*G),$$

where the colimit runs over all cocompact subsets $X \subset \underline{E}G$. It is not difficult to see that this composite is indeed natural with respect to inclusions of proper G-spaces.

Mishchenko-Fomenko Index Theory

In [2] Mishchenko and Fomenko developed the theory of differential operators acting not on sections of a smooth vector bundle over a smooth manifold, but on sections of so-called Hilbert-A-module bundles, where A is some C^*-algebra. These are smooth locally trivial bundles whose fibers are finitely generated projective A-modules.
There is a K-theory group $K(X;A)$ associated to these kinds of bundles and we have the following

Proposition. *Let X be a compact space. Then there is an isomorphism*

$$K(X;A) \xrightarrow{\cong} K_0(C(X) \otimes A)$$

induced by assigning to such a Hilbert-A-module bundle the space of sections, which is a module over the sections of the trivial bundle.

The main work of [2] is to develop an index theory of differential operators acting on sections of a smooth Hilbert-A-module bundle. The first step is (as in the usual theory) to extend a differential operator to a bounded operator between Sobolev completions of the spaces of sections. There is a notion of socalled A-Fredholm operators and the result is that any A-Fredholm operator D has an index $\mathrm{ind}(D) \in K_0(A)$. Moreover there are many examples of A-Fredholm operators that are constructed out of elliptic differential operators on an ordinary vector bundle which are twisted by a Hilbert-A-module bundle (here one needs an appropriate version of the Rellich Lemma and the notion of a connection on Hilbert-A-module bundles).
This is related to the analytical assembly map as follows. In [3], socalled *geometric* K-homology groups $K_*^{\mathrm{geo}}(X)$ are defined as the free abelian group generated by triples (M, f, E) where

- M is a smooth spin^c manifold,
- $f : M \to X$ is a continuous map, and
- E is a smooth hermitian bundle over M

subject to some relations like *bordism* and *vector bundle modification.*
There is a natural transformation

$$\begin{aligned} \eta_X : K_*^{\mathrm{geo}}(X) &\longrightarrow KK_*(C_0(X), \mathbb{C}) \\ [M,f,E] &\longmapsto [f_*(D_E)] \end{aligned}$$

and it is proven in [3] that this is an isomorphism provided that X is compact.
We can now define a map

$$K_0^{\mathrm{geo}}(BG) \xrightarrow{\mathrm{MF}} K_0(C_r^*G)$$

by the following construction. We take an element $[M, f, E] \in K_*^{\mathrm{geo}}(BG)$. We let $\mathcal{L}_{BG} = EG \times_G C_r^*G$ be the Mishchenko line bundle (G acts by left multiplication on C_r^*G). We can twist the Dirac operator D_E over M with the Mishchenko line bundle $\mathcal{L}_{BG}$ and obtain a C_r^*G-Fredholm operator $\mathcal{D}$ which has an index in $K_0(C_r^*G)$ and we define $\mathrm{MF}[M, f, E] = \mathrm{ind}(\mathcal{D}) \in K_0(C_r^*G)$.
There is a KK-theoretic interpretation given as follows. In [4] it is shown that the map MF coincides with the composite

$$K_0^{\mathrm{geo}}(BG) \xrightarrow{\eta_{BG}} RKK(C_0(BG), \mathbb{C}) \xrightarrow{\tau_{C_r^*G}} RKK(C_0(BG) \otimes C_r^*G, C_r^*G) \xrightarrow{-\circ[\mathcal{L}_{BG}]} KK(\mathbb{C}, C_r^*G)$$

which we call the *generalized Mishchenko-Fomenko Index*. Note that this result implies that the previous ad hoc construction of assigning to a geometric cycle its equivariant index is well-defined, which is not clear a priori.

The Comparison Theorem

In a recent result I was able to prove the following

Theorem. *Suppose that the group G is torsionfree. Then there is an identification of the domains of the analytical assembly map and the generalized Mishchenko-Fomenko Index such that the diagram*

$$\begin{tikzcd} RK_*(BG) \arrow[r, "\mathrm{MF}"] & KK(\mathbb{C}, C_r^*G) \\ RK_*^G(\underline{E}G) \arrow[u, "\cong"] \arrow[ur, bend right, "\mathcal{A}"'] & \end{tikzcd}$$

commutes.

We note that by assumption $\underline{E}G = EG$. The identification of the domains proceeds in two steps. Firstly we have the following theorem due to Green-Julg.

Proposition. *For any cocompact G-space X we have a canonical isomorphism*

$$\mathrm{GJ} : KK_*^G(C_0(X), \mathbb{C}) \longrightarrow KK_*(C_0(X) \rtimes G, \mathbb{C}),$$

i.e., the equivariant K-homology is isomorphic to the unequivariant K-homology of the full crossed product.

In the case of free actions, the crossed product algebra is up to KK-equivalence a quotient, as the following proposition by Green shows.

Proposition. *Suppose X is a proper and free G-space. Then the algebras $C_0(X) \rtimes G$ and $C_0(X/G)$ are Morita equivalent.*

It follows that there is an invertible class $[\mathcal{F}(X)] \in KK(C_0(X/G), C_0(X) \rtimes G)$. If furthermore the space X is cocompact then under the inclusion of scalars $i : \mathbb{C} \longrightarrow C(X/G)$ the element $[\mathcal{F}(X)]$ restricts to $[p_X] \in KK(\mathbb{C}, C_0(X) \rtimes G)$.
Using generalized fixed point constructions one can define an element $\widetilde{L}_X \in KK(C_0(X/G), C_0(X/G) \otimes C_r^*G)$ which restricts in the cocompact case to the Mishchenko line bundle $[\mathcal{L}_{BG}] \in KK(\mathbb{C}, C(X/G) \otimes C_r^*G)$. This is done as follows. Consider the following two G-C^*-algebras $A = (C_0(X) \otimes C_r^*G, \tau \otimes \mathrm{ad}_\lambda)$ and $B = (C_0(X) \otimes C_r^*G, \tau \otimes \mathrm{id})$. The module $\mathcal{E} = (C_0(X) \otimes C_r^*G, \tau \otimes \lambda)$ is an equivariant imprimitivity-A-B-bimodule. The generalized fixed point constructions A^G, B^G and $\mathcal{E}^G$ in this case are simply the sections of the bundle $X \times_G C_r^*G$ where the G-actions on C_r^*G are as in the notation of the algebras. In particular the module $\mathcal{E}^G$ is just the module of sections of the Mishchenko line bundle. To obtain the element $\widetilde{L}_X$ we simply need to note that there is an inclusion $j : C_0(X/G) \longrightarrow A^G$ and restrict the element $[\mathcal{E}^G] \in KK(A^G, B^G)$ along this morphism.

The Proof Continued

We can now consider the following diagram for a proper and cocompact G-space X:

$$\begin{tikzcd} & & KK_*(\mathbb{C}, C_r^*G) \\ KK_*(C(X/G), \mathbb{C}) \arrow[r, "\tau_{C_r^*G}"] & KK_*(C(X/G) \otimes C_r^*G, C_r^*G) \arrow[r, "-\circ\widetilde{L}_X"] \arrow[ur, bend left, "-\circ[\mathcal{L}_{X/G}]"] \arrow[r, phantom, shift left=6, "(4)"] & KK_*(C(X/G), C_r^*G) \arrow[u, "i^*"'] \\ KK_*(C_0(X) \rtimes G, \mathbb{C}) \arrow[u, "-\circ[\mathcal{F}(X)]"] \arrow[r, "\tau_{C_r^*G}"] \arrow[ur, phantom, "(2)"] & KK_*(C_0(X) \rtimes G \otimes C_r^*G, C_r^*G) \arrow[u, "-\circ\tau_{C_r^*G}[\mathcal{F}(X)]"'] \arrow[r, "\Delta^*"] \arrow[ur, phantom, "(3)"] & KK_*(C_0(X) \rtimes G, C_r^*G) \arrow[u, "-\circ[\mathcal{F}(X)]"'] \arrow[uu, bend right=60, "-\circ[p_X]"', "(5)"] \\ KK_*^G(C_0(X), \mathbb{C}) \arrow[u, "\mathrm{GJ}"] \arrow[urr, bend right, "j_r^G"'] \arrow[ur, phantom, "(1)"] & & \end{tikzcd}$$

Taking a colimit over all cocompact $X \subset EG$ we see that the top horizontal composite is the generalized Mishchenko-Fomenko Index, the left vertical composite is the identification of the domains and the lower horizontal composite is the analytical assembly map.
The fact that the subdiagram (2) commutes is a classical fact, (4) and (5) commute by construction of the elements $[\mathcal{F}(X)]$ and $\widetilde{L}_X$. It is a Lemma that diagram (1) commutes, but not too complicated. The real problem lies in showing that diagram (3) commutes, but using some recent results of Buss-Echterhoff we were able to conclude this as well.

References

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