



Analytical Assembly and Index Theory

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The Baum-Connes Conjecture à la Kasparov

Let G be a countable discrete group. There is a map called the *analytical assembly map*

$$\mathcal{A} : RK_*^G(\underline{EG}) \longrightarrow K_*(C_r^*G)$$

between the G -equivariant compactly supported analytic K -homology of the classifying space for proper actions and the K -theory of the reduced group C^* -algebra. We recall that a classifying space for proper actions is a G -CW-complex whose H -fixed points are contractible if H is a finite subgroup of G and are empty else. Such a space is automatically a terminal object in the full subcategory of the equivariant homotopy category consisting of proper G -spaces.

The analytic assembly map turns up in the Baum-Connes conjecture:

Conjecture. For every countable discrete group G the analytical assembly map

$$\mathcal{A} : RK_*^G(\underline{EG}) \xrightarrow{\cong} K_*(C_r^*G)$$

is an isomorphism of abelian groups.

There are many definitions of the assembly map and one of my goals is to show that all these constructions indeed coincide. Here I will only talk about two descriptions - one purely in terms of KK -theory and an index-theoretic description. The following (analytical) construction of the assembly map is the approach proposed by Kasparov, and is the one used when applying the so-called Dirac-dual-Dirac method (a strategy of proving the Baum-Connes conjecture). To define this map we need the following

Lemma. Let X be a proper and cocompact G -space. Then there exists a function $\psi \in C_c(X)$ such that for all $x \in X$ we have $\sum_{g \in G} \psi(gx) = 1$. Moreover the element

$$p_X(g, x) = \sqrt{\psi(x) \cdot \psi(g^{-1}x)} \in C_0(X) \rtimes G$$

is a projection and its K -theory class $[p_X] \in KK(\mathbb{C}, C_0(X) \rtimes G)$ is independent of the function ψ .

The following proposition is a classical fact in KK -theory and was first formulated and proven in [1].

Proposition. There is a functor $j_r^G : KK^G \longrightarrow KK$ from the equivariant KK -category to the unequivariant KK -category, called the reduced descent homomorphism. On objects this functor is given by the reduced crossed product functor, i.e., for any G - C^* -algebra A we have

$$j_r^G(A) = A \rtimes_r G.$$

Sometimes reduced and full crossed products by G coincide. A result concerning this is the following

Proposition. If X is a proper G -space then $C_0(X) \rtimes G = C_0(X) \rtimes_r G$. Moreover this algebra is nuclear.

Using this we can now define the analytical assembly map by the map induced on colimits of the following composite

$$KK_*^G(C_0(X), \mathbb{C}) \xrightarrow{j_r^G} KK_*(C_0(X) \rtimes G, C_r^*G) \xrightarrow{-\circ[p_X]} KK_*(\mathbb{C}, C_r^*G),$$

where the colimit runs over all cocompact subsets $X \subset \underline{EG}$. It is not difficult to see that this composite is indeed natural with respect to inclusions of proper G -spaces.

Mishchenko-Fomenko Index Theory

In [2] Mishchenko and Fomenko developed the theory of differential operators acting not on sections of a smooth vector bundle over a smooth manifold, but on sections of so-called Hilbert- A -module bundles, where A is some C^* -algebra. These are smooth locally trivial bundles whose fibers are finitely generated projective A -modules.

There is a K -theory group $K(X; A)$ associated to these kinds of bundles and we have the following

Proposition. Let X be a compact space. Then there is an isomorphism

$$K(X; A) \xrightarrow{\cong} K_0(C(X) \otimes A)$$

induced by assigning to such a Hilbert- A -module bundle the space of sections, which is a module over the sections of the trivial bundle.

The main work of [2] is to develop an index theory of differential operators acting on sections of a smooth Hilbert- A -module bundle. The first step is (as in the usual theory) to extend a differential operator to a bounded operator between Sobolev completions of the spaces of sections. There is a notion of so-called A -Fredholm operators and the result is that any A -Fredholm operator D has an index $\text{ind}(D) \in K_0(A)$. Moreover there are many examples of A -Fredholm operators that are constructed out of elliptic differential operators on an ordinary vector bundle which are twisted by a Hilbert- A -module bundle (here one needs an appropriate version of the Rellich Lemma and the notion of a connection on Hilbert- A -module bundles).

This is related to the analytical assembly map as follows. In [3], so-called *geometric K -homology groups* $K_*^{\text{geo}}(X)$ are defined as the free abelian group generated by triples (M, f, E) where

- M is a smooth spin^c manifold,
- $f : M \rightarrow X$ is a continuous map, and
- E is a smooth hermitian bundle over M

subject to some relations like *bordism* and *vector bundle modification*.

There is a natural transformation

$$\eta_X : K_*^{\text{geo}}(X) \longrightarrow KK_*(C_0(X), \mathbb{C})$$

$$[M, f, E] \longmapsto [f_*(D_E)]$$

and it is proven in [3] that this is an isomorphism provided that X is compact.

We can now define a map

$$K_0^{\text{geo}}(BG) \xrightarrow{\text{MF}} K_0(C_r^*G)$$

by the following construction. We take an element $[M, f, E] \in K_*^{\text{geo}}(BG)$. We let $\mathcal{L}_{BG} = EG \times_G C_r^*G$ be the Mishchenko line bundle (G acts by left multiplication on C_r^*G). We can twist the Dirac operator D_E over M with the Mishchenko line bundle \mathcal{L}_{BG} and obtain a C_r^*G -Fredholm operator \mathcal{D} which has an index in $K_0(C_r^*G)$ and we define $\text{MF}[M, f, E] = \text{ind}(\mathcal{D}) \in K_0(C_r^*G)$.

There is a KK -theoretic interpretation given as follows. In [4] it is shown that the map MF coincides with the composite

$$K_0^{\text{geo}}(BG) \xrightarrow{\eta_{BG}} RKK(C_0(BG), \mathbb{C}) \xrightarrow{\tau_{C_r^*G}} RKK(C_0(BG) \otimes C_r^*G, C_r^*G) \xrightarrow{-\circ[\mathcal{L}_{BG}]} KK(\mathbb{C}, C_r^*G)$$

which we call the *generalized Mishchenko-Fomenko Index*. Note that this result implies that the previous ad hoc construction of assigning to a geometric cycle its equivariant index is well-defined, which is not clear a priori.

The Comparison Theorem

In a recent result I was able to prove the following

Theorem. Suppose that the group G is torsionfree. Then there is an identification of the domains of the analytical assembly map and the generalized Mishchenko-Fomenko Index such that the diagram

$$\begin{array}{ccc} RK_*(BG) & \xrightarrow{\text{MF}} & KK(\mathbb{C}, C_r^*G) \\ \cong \uparrow & \nearrow \mathcal{A} & \\ RK_*^G(\underline{EG}) & & \end{array}$$

commutes.

We note that by assumption $\underline{EG} = EG$. The identification of the domains proceeds in two steps. Firstly we have the following theorem due to Green-Julg.

Proposition. For any cocompact G -space X we have a canonical isomorphism

$$\text{GJ} : KK_*^G(C_0(X), \mathbb{C}) \longrightarrow KK_*(C_0(X) \rtimes G, \mathbb{C}),$$

i.e., the equivariant K -homology is isomorphic to the unequivariant K -homology of the full crossed product.

In the case of free actions, the crossed product algebra is up to KK -equivalence a quotient, as the following proposition by Green shows.

Proposition. Suppose X is a proper and free G -space. Then the algebras $C_0(X) \rtimes G$ and $C_0(X/G)$ are Morita equivalent.

It follows that there is an invertible class $[\mathcal{F}(X)] \in KK(C_0(X/G), C_0(X) \rtimes G)$. If furthermore the space X is cocompact then under the inclusion of scalars $i : \mathbb{C} \longrightarrow C(X/G)$ the element $[\mathcal{F}(X)]$ restricts to $[p_X] \in KK(\mathbb{C}, C_0(X) \rtimes G)$.

Using generalized fixed point constructions one can define an element $\tilde{L}_X \in KK(C_0(X/G), C_0(X/G) \otimes C_r^*G)$ which restricts in the cocompact case to the Mishchenko line bundle $[\mathcal{L}_{BG}] \in KK(\mathbb{C}, C(X/G) \otimes C_r^*G)$. This is done as follows. Consider the following two G - C^* -algebras $A = (C_0(X) \otimes C_r^*G, \tau \otimes \text{ad}_\lambda)$ and $B = (C_0(X) \otimes C_r^*G, \tau \otimes \text{id})$. The module $\mathcal{E} = (C_0(X) \otimes C_r^*G, \tau \otimes \lambda)$ is an equivariant imprimitivity- A - B -bimodule. The generalized fixed point constructions A^G , B^G and \mathcal{E}^G in this case are simply the sections of the bundle $X \times_G C_r^*G$ where the G -actions on C_r^*G are as in the notation of the algebras. In particular the module \mathcal{E}^G is just the module of sections of the Mishchenko line bundle. To obtain the element \tilde{L}_X we simply need to note that there is an inclusion $j : C_0(X/G) \longrightarrow A^G$ and restrict the element $[\mathcal{E}^G] \in KK(A^G, B^G)$ along this morphism.

The Proof Continued

We can now consider the following diagram for a proper and cocompact G -space X :

$$\begin{array}{ccccc} & & & & KK_*(\mathbb{C}, C_r^*G) \\ & & & \nearrow -\circ[\mathcal{L}_{X/G}] & \uparrow i^* \\ KK_*(C(X/G), \mathbb{C}) & \xrightarrow{\tau_{C_r^*G}} & KK_*(C(X/G) \otimes C_r^*G, C_r^*G) & \xrightarrow{-\circ\tilde{L}_X} & KK_*(C(X/G), C_r^*G) \\ \uparrow -\circ[\mathcal{F}(X)] & & \uparrow -\circ\tau_{C_r^*G}[\mathcal{F}(X)] & & \uparrow -\circ[\mathcal{F}(X)] \\ KK_*(C_0(X) \rtimes G, \mathbb{C}) & \xrightarrow{\tau_{C_r^*G}} & KK_*(C_0(X) \rtimes G \otimes C_r^*G, C_r^*G) & \xrightarrow{\Delta^*} & KK_*(C_0(X) \rtimes G, C_r^*G) \\ \uparrow \text{GJ} & & \uparrow (1) & & \uparrow j_r^G \\ KK_*^G(C_0(X), \mathbb{C}) & & & & \end{array}$$

Taking a colimit over all cocompact $X \subset \underline{EG}$ we see that the top horizontal composite is the generalized Mishchenko-Fomenko Index, the left vertical composite is the identification of the domains and the lower horizontal composite is the analytical assembly map.

The fact that the subdiagram (2) commutes is a classical fact, (4) and (5) commute by construction of the elements $[\mathcal{F}(X)]$ and \tilde{L}_X . It is a Lemma that diagram (1) commutes, but not too complicated. The real problem lies in showing that diagram (3) commutes, but using some recent results of Buss-Echterhoff we were able to conclude this as well.

References

- [1] Gennadi Kasparov, *Equivariant KK -Theory and the Novikov Conjecture*, Invent. math. 91,147-201, 1988.
- [2] A.S Mishchenko, A.T. Fomenko, *The Index of Elliptic Operators over C^* -Algebras*, English Translation: Math. USSR-Izv. 15, no. 1,87-112, 1980.
- [3] P. Baum, A. Connes, T. Schick, *On the Equivalence of Geometric and Analytic K -Homology*, arXiv:math/0701484v4, 2009.
- [4] T. Schick, *L^2 -Index Theorems, KK -Theory and Connections*, arXiv:math/0306171v4, 2005.
- [5] M. Land, *Assembly Maps*, Master's thesis, www.math.uni-bonn.de/people/land, 2012.